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# On maximal antihierarchic sets of integers

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*Abstract*

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We write the nonnegative integers in a fixed base $b \geqslant 2$, and call two such integers $c$ and $d$ comparable if each digit of $c$ is at least as big as the respective digit of $d$, or vice versa; otherwise we say that $c$ and $d$ are uncomparable. A set of pairwise uncomparable integers will be called antihierarchic.

The paper contains explicit and asymptotic formulae as well as upper and lower bounds for the cardinality of maximal antihierarchic sets $S \subseteq \{1, 2, \ldots, n\}$ for given $n$.

## 1. Introduction

It was asked in [4] to give a proof for the fact that any subset $S$ of the positive integers less than 1000, satisfying $|S| \geqslant 76$, contains two elements $c \neq d$, such that the subtraction $c-d$ resp. $d-c$ written in base 10 is done without 'borrowing', i.e. each digit of $c$ is at least as big as the respective digit of $d$, or vice versa. In this case we will call $c$ and $d$ comparable, otherwise uncomparable. A set of pairwise uncomparable integers is called antihierarchic.

In this paper we will give explicit and asymptotic formula as well as upper and lower bounds for the cardinality of maximal antihierarchic sets of integers written in arbitrary base and with an arbitrary number of digits, using combinatorial methods including Sperner theory and generating functions.

For integers $b \geqslant 2$ and $m \geqslant 0$ let $m_b$ denote $m$ written in base $b$. For a positive integer $t$ let $A(t) = \{0, 1, \ldots, t\}$. We define for a positive integer $N$

$$M_b(N) = \max |\{S \subseteq A(N-1)\colon m_b, m'_b \text{ are uncomparable for all } m \neq m' \text{ in } S\}|;$$

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especially, set for $n \in \mathbb{N}$

$$M(b;n) = M_b(b^n).$$

By what was said above, we have for instance $M(10;3) = 75$.

**Theorem 1.** *Let $b \geqslant 2$ and $n \geqslant 1$ be integers; let $h = [n(b-1)/2]$. Then*

$$M(b;n) = \sum_{0 \leqslant j \leqslant h/b} (-1)^j \binom{n}{j} \binom{n-1+h-bj}{n-1}.$$

In general, the absolute values of the summands above do not form a monotonic sequence. Therefore, one cannot deduce upper or lower bounds for $M(b;n)$ from Theorem 1 in a simple fashion. However, the number of terms is of logarithmic order compared with the cardinality of $A(b^n - 1)$, which makes the actual computation rather easy; for example, $M(10;3) = 105 - 30 = 75$.

**Theorem 2.** *Let $b \geqslant 2$ and $n \geqslant 1$ be integers. Then*

$$M(b;n) = \frac{2}{\pi} \int_0^{\pi/2} \left( \frac{\sin bx}{\sin x} \right)^n g_{b,n}(x) \, \mathrm{d}x, \tag{1}$$

*where*

$$g_{b,n}(x) = \begin{cases} 1 & \text{for } 2 \mid n(b-1), \\ \cos x & \text{otherwise.} \end{cases}$$

*For fixed $b$ and $n \to \infty$, we have*

$$M(b;n) = \left( \frac{6}{\pi(b^2-1)} \right)^{1/2} b^n n^{-1/2} + \mathrm{O}(b^n n^{-3/2}), \tag{2}$$

*where the implied constant in* O( ) *only depends on $b$.*

**Theorem 3.** *Let $b \geqslant 2$ be fixed and $\varepsilon > 0$. Then for sufficiently large $N$*

$$(\tfrac{1}{2}C - \varepsilon) \frac{N}{(\log N)^{1/2}} < M_b(N) < (2C + \varepsilon) \frac{N}{(\log N)^{1/2}},$$

*where*

$$C = \left( \frac{6 \log b}{\pi(b^2-1)} \right)^{1/2}$$

## 2. Sperner theorems

(For terminology and well-known results used in this paragraph see for instance [1].)

Let $A(t)$ be given the natural order, i.e. $0<1<\cdots<t$. Then, for any $n \in \mathbb{N}$, the product space $A(t)^n$ can be partially ordered by

$$\boldsymbol{a} \leqslant \boldsymbol{a}' \Leftrightarrow a_i \leqslant a_i' \ (i=1,\ldots,n)$$

for all $\boldsymbol{a}=(a_1,\ldots,a_n)$, $\boldsymbol{a}'=(a_1',\ldots,a_n') \in A(t)^n$. We obviously have the following.

**Lemma 1.** *For $a$, $a' \in A(b^n-1)$, $a \neq a'$, let $a=a_1+a_2b+\cdots+a_nb^{n-1}$, $a'=a_1'+a_2'b+\cdots+a_n'b^{n-1}$ with $\boldsymbol{a}=(a_1,\ldots,a_n)$, $\boldsymbol{a}'=(a_1',\ldots,a_n') \in A(b-1)^n$. Then*

$$a \text{ and } a' \text{ are comparable} \Leftrightarrow \boldsymbol{a}<\boldsymbol{a}' \text{ or } \boldsymbol{a}'<\boldsymbol{a}.$$

There is a unique minimal element in $A(t)^n$, namely $(0,\ldots,0)$. Also all maximal chains between two fixed elements of $A(t)^n$ have the same length. Therefore, there is a rank function $r: A(t)^n \to \mathbb{N}_0$, given by

$$r(\boldsymbol{a})=\sum_{1\leqslant i\leqslant n} a_i, \tag{3}$$

which is the length of a maximal chain between $(0,\ldots,0)$ and $\boldsymbol{a}$. Define the $l$-level $N_l(A(t)^n)=\{\boldsymbol{a}\in A(t)^n\colon\ r(\boldsymbol{a})=l\}$, and the $l$th level number $W_l=W_l(A(t)^n)=|N_l(A(t)^n)|$. Finally, let $r_{\max}=r_{\max}(A(t)^n)$ be the rank of $A(t)^n$, i.e. $r_{\max}=\max\{r(\boldsymbol{a})\colon \boldsymbol{a}\in A(t)^n\}$. It is easily seen that

$$r_{\max}(A(t)^n)=nt. \tag{4}$$

The following clearly holds.

**Lemma 2.** *For $0\leqslant l\leqslant r_{\max}$*

$$W_l(A(t)^n)=W_{r_{\max}-l}(A(t)^n).$$

Let $s(A(t)^n)$ denote the cardinality of a maximal antichain in $A(t)^n$. $s$ is called the Sperner number of $A(t)^n$.

**Proposition 1.**

$$s(A(t)^n)=\left|\left\{\boldsymbol{a}\in A(t)^n\colon \sum_{1\leqslant i\leqslant n} a_i=\left[\frac{nt}{2}\right]\right\}\right|.$$

**Proof.** (For definitions and theorems used in this proof see [1, Chapt. VIII, 3].)

$A(t)^n$ is a finite chain product which by a theorem of De Bruijn et al. is symmetric. Thus it possesses the unimodular property, which means there is a $j$, $0\leqslant j\leqslant r_{\max}$, such that $W_0\leqslant W_1\leqslant\cdots\leqslant W_j\geqslant W_{j+1}\geqslant\cdots\geqslant W_{r_{\max}}$. Using Lemma 2 and (4), we get

$$\max_{0\leqslant l\leqslant r_{\max}} W_l=W_{[r_{\max}/2]}=W_{[nt/2]}. \tag{5}$$

Since $A(t)^n$ is symmetric, it also has the Sperner property, which by definition means that $s(A(t)^n) = \max_l W_l$. By (5), (3) and the definition of $W_l$, this proves the proposition. □

Lemma 1 and Proposition 1 immediately give the following.

**Corollary 1.** *For $b \geq 2$ and $n \geq 1$,*

$$M(b;n) = \left|\left\{ \boldsymbol{a} \in A(b-1)^n : \sum_{1 \leq i \leq n} a_i = \left[\frac{n(b-1)}{2}\right]\right\}\right|.$$

**Corollary 2.** *For $b \geq 2$ and $n \geq 1$, an antihierarchic set $S \subseteq A(b^n - 1)$ is given by*

$$S = \left\{ m < b^n : \text{sum of digits of } m_b = \left[\frac{n(b-1)}{2}\right]\right\}.$$

## 3. Generating functions: proof of Theorem 1

For nonnegative integers $k$, $n$ and $t \geq 1$ let

$$\binom{n}{k}_t = \left|\left\{ \boldsymbol{a} \in A(t)^n : \sum_{1 \leq i \leq n} a_i = k\right\}\right|.$$

Expanding

$$f_{n,t}(x) = (1 + x + x^2 + \cdots + x^t)^n \tag{6}$$

into a power series $\sum a_k x^k$, we obviously have

$$a_k = \binom{n}{k}_t.$$

Therefore, we get the following.

**Proposition 2.**

$$\binom{n}{k}_t = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \binom{n-1+k-(t+1)j}{n-1}.$$

Combining Proposition 2 and Corollary 1 immediately yields Theorem 1.

**Remark.** The numbers $\binom{n}{k}_t$ are called polynomial coefficients in [3]; they have been studied by André [2], and in a much more general setting by Montel [5]. Since they satisfy the recursion formula

$$\binom{n+1}{k}_t = \sum_{m=0}^{k} \binom{n}{k-m}_t, \qquad \binom{0}{k}_t = \begin{cases} 1 & \text{for } k = 0, \\ 0 & \text{otherwise,} \end{cases}$$

which is easily deduced from their generating function (6), they may be regarded as generalized binomial coefficients. In fact, for $t=1$ they are the ordinary binomial coefficients. For arbitrary $t$ the polynomial coefficients share a lot of properties with the ordinary binomial coefficients (see [3, p. 77–78]).

## 4. Proof of Theorem 2

Applying Cauchy's integral formula to the generating function $f_{n,t}$ in (6), we get by (7) and Corollary 1

$$\binom{n}{k}_t = \frac{1}{2\pi i}\int_{\mathscr{C}} f_{n,t}(z) z^{-k-1}\,\mathrm{d}z,$$

where $\mathscr{C}$ denotes a counterclockwise oriented circle of radius 1 centered at the origin. By substituting $z = \mathrm{e}^{2xi}$, $0 \leq x \leq \pi$, we get

$$\begin{aligned}\binom{n}{k}_t &= \frac{1}{\pi}\int_0^{\pi}\left(\frac{1-\mathrm{e}^{2(t+1)xi}}{1-\mathrm{e}^{2xi}}\right)^n \mathrm{e}^{-2kxi}\,\mathrm{d}x\\ &= \frac{1}{\pi}\int_0^{\pi}\left(\frac{\sin((t+1)x)}{\sin x}\right)^n \mathrm{e}^{(tn-2k)xi}\,\mathrm{d}x\\ &= \frac{1}{\pi}\int_0^{\pi}\left(\frac{\sin((t+1)x)}{\sin x}\right)^n \cos((tn-2k)x)\,\mathrm{d}x\\ &= \frac{2}{\pi}\int_0^{\pi/2}\left(\frac{\sin((t+1)x)}{\sin x}\right)^n \cos((tn-2k)x)\,\mathrm{d}x.\end{aligned}$$

The last equality follows from the symmetry of the integrand with respect to $x=\pi/4$. Choosing $t=b-1$ and $k=[n(b-1)/2]$, Corollary 1 implies (1).

The remainder of this paragraph is devoted to a proof of (2). From now on let $b \geq 2$ be a fixed integer. We have

$$\begin{aligned}\left|\int_{2/b}^{\pi/2}\left(\frac{\sin bx}{\sin x}\right)^n g_{b,n}(x)\,\mathrm{d}x\right| &\leq \left(\sin\frac{2}{b}\right)^{-n}\int_0^{\pi/2}\mathrm{d}x\\ &\leq \frac{\pi}{2}\left(\frac{3b}{5}\right)^n.\end{aligned}$$

In order to prove (2) it therefore suffices by (i) to show that

$$\int_0^{2/b}\left(\frac{\sin bx}{b\sin x}\right)^n g_{b,n}(x)\,\mathrm{d}x = \frac{1}{2}\left(\frac{\pi}{Bn}\right)^{1/2} + \mathrm{O}(n^{-3/2}), \tag{8}$$

where $B = \frac{1}{6}(b^2-1)$.

Looking at the power series expansion of $\sin x/x$, it is easily seen, that for any $x$, $0<x\leq 1$, there is a $\gamma_1 = \gamma_1(x)$ with $0<\gamma_1 \leq 1/18$, such that

$$\frac{x}{\sin x} = 1 + \tfrac{1}{6}x^2 + \gamma_1 x^4. \tag{9}$$

By the same argument, one finds for any $x \in I_b(0, 2/b)$ a $\gamma_2 = \gamma_2(x)$, $0 < \gamma_2 \leq 1/120$, satisfying

$$\frac{\sin bx}{bx} = 1 - \tfrac{1}{6}(bx)^2 + \gamma_2(bx)^4. \tag{10}$$

Combining (9) and (10), we can show that for any $x \in I_b$ there exists a $\gamma_3 = \gamma_3(x)$ with

$$|\gamma_3| \leq \tfrac{1}{15} + \tfrac{1}{120}b^4 \tag{11}$$

and

$$\frac{\sin bx}{b \sin x} = 1 - Bx^2 - \gamma_3 x^4. \tag{12}$$

By (11), we obviously have for $x \in I_b$

$$0 < Bx^2 + \gamma_3 x^4 < \tfrac{3}{5}. \tag{13}$$

For $0 < y < \frac{3}{5}$,

$$y + \frac{y^2}{2} \leq \sum_{j=1}^{\infty} \frac{y^j}{j} \leq y + \frac{y^2}{2} \sum_{j=0}^{\infty} y^j = y + \frac{y^2}{2}\frac{1}{1-y} \leq y + \tfrac{5}{4}y^2,$$

hence there is a $\gamma_4 = \gamma_4(y)$ with $\frac{1}{2} \leq \gamma_4 \leq \frac{5}{4}$ and

$$\log(1 - y) = -y - \gamma_4 y^2.$$

By (13) we may apply this to (12). Thus there exists a constant $c_1$ depending only on $b$, such that for any $x \in I_b$, there are $\gamma_5 = \gamma_5(x)$ and $\gamma_6 = \gamma_6(x)$, $\max(|\gamma_5|, |\gamma_6|) < c_1$, satisfying

$$\begin{aligned}\left(\frac{\sin bx}{b \sin x}\right)^n &= \exp(n \log(1 - (Bx^2 + \gamma_3 x^4))) \\ &= \exp(-n(Bx^2 + \gamma_5 x^4)) \\ &= e^{-Bnx^2}(1 + \gamma_6 n x^4).\end{aligned} \tag{14}$$

Notice that the bounds for $\gamma_1$, $\gamma_2$ and $\gamma_3$ have to be explicit, since (13) is essential for the convergence of the power series of $\log(1 - y)$.

By looking at the power series of $\cos x$ it is obvious that for $x \in I_b$ there is a $\gamma_7 = \gamma_7(x)$ with $|\gamma_7| \leq \frac{1}{2}$ and

$$g_{b,n}(x) = 1 - \gamma_7 x^2, \tag{15}$$

where $g_{b,n}$ was defined in (1). By (14) and (15), there exists a constant $c_2$ only depending on $b$, such that for any $x \in I_b$ there are $\gamma_8 = \gamma_8(x)$ and $\gamma_9 = \gamma_9(x)$ satisfying

$$\max(|\gamma_8|, |\gamma_9|) < c_2 \tag{16}$$

and

$$\left(\frac{\sin bx}{b\sin x}\right)^{n} g_{b,n}(x) = \mathrm{e}^{-Bnx^2}(1+\gamma_8 x^2+\gamma_9 n x^4). \tag{17}$$

We have

$$\begin{aligned}\int_0^{2/b} \mathrm{e}^{-Bnx^2}\,\mathrm{d}x &= \int_0^{\infty} \mathrm{e}^{-Bnx^2}\,\mathrm{d}x - \int_{2/b}^{\infty} \mathrm{e}^{-Bnx^2}\,\mathrm{d}x\\ &= \frac{1}{2}\left(\frac{\pi}{Bn}\right)^{1/2} + \mathrm{O}\left(n^{-1/2}\int_{c_3 n}^{\infty} \mathrm{e}^{-t}t^{-1/2}\,\mathrm{d}t\right),\end{aligned}$$

where the constant $c_3$ and the implicit constant in O( ) may only depend on $b$. Hence

$$\begin{aligned}\int_0^{2/b} \mathrm{e}^{-Bnx^2}\,\mathrm{d}x &= \frac{1}{2}\left(\frac{\pi}{Bn}\right)^{1/2} + \mathrm{O}\left(\frac{1}{n}\int_{c_3 n}^{\infty} \mathrm{e}^{-t}\,\mathrm{d}t\right)\\ &= \frac{1}{2}\left(\frac{\pi}{Bn}\right)^{1/2} + \mathrm{O}\left(\frac{1}{n}\mathrm{e}^{-c_3 n}\right).\end{aligned} \tag{18}$$

By (16), we have

$$\begin{aligned}\int_0^{2/b} \mathrm{e}^{-Bnx^2}\gamma_8(x)x^2\,\mathrm{d}x &\le c_2\int_0^{\infty} \mathrm{e}^{-Bnx^2}x^2\,\mathrm{d}x\\ &\ll n^{-3/2}\int_0^{\infty} \mathrm{e}^{-t}t^{1/2}\,\mathrm{d}t.\end{aligned} \tag{19}$$

Again the implicit constant may only depend on $b$. By partial integration, we get

$$\begin{aligned}\int_0^{\infty} \mathrm{e}^{-t}t^{1/2}\,\mathrm{d}t &= \int_0^{1} \mathrm{e}^{-t}t^{1/2}\,\mathrm{d}t + \int_1^{\infty} \mathrm{e}^{-t}t^{1/2}\,\mathrm{d}t\\ &\ll 1 + \int_1^{\infty} \mathrm{e}^{-t}t^{-1/2}\,\mathrm{d}t \ll 1.\end{aligned} \tag{20}$$

Therefore,

$$\int_0^{2/b} \mathrm{e}^{-Bnx^2}\gamma_8(x)x^2\,\mathrm{d}x \ll n^{-3/2}. \tag{21}$$

Similar arguments as in (19) and (20) yield

$$\int_0^{2/b} \mathrm{e}^{-Bnx^2}\gamma_9(x)x^4\,\mathrm{d}x \ll n^{-5/2}. \tag{22}$$

Applying (18) and the upper bounds (21) and (22) to (17), we finally get (8). This completes the proof of Theorem 2. □

**Corollary 3.** *Let $b \geqslant 2$ be fixed and $\varepsilon > 0$. Then for sufficiently large $N$*

$$\left(\frac{1}{b}C - \varepsilon\right)\frac{N}{(\log N)^{1/2}} < M_b(N) < (bC + \varepsilon)\frac{N}{(\log N)^{1/2}}, \tag{23}$$

*where*

$$C = \left(\frac{6\log b}{\pi(b^2-1)}\right)^{1/2}$$

**Proof.** Choose $n$ such that

$$b^n \leqslant N < b^{n+1}. \tag{24}$$

The sequence $M_b(N)$ is obviously increasing. Hence

$$M(b; n) \leqslant M_b(N) \leqslant M(b; n+1).$$

By Theorem 2 and (24), we have for sufficiently large $N$, i.e. sufficiently large $n$,

$$\begin{aligned} M(b;n) &> ((B\pi)^{-1/2} - \varepsilon)b^n n^{-1/2} \\ &\geqslant ((B\pi)^{-1/2} - \varepsilon)\frac{N}{b}\left(\frac{\log N}{\log b}\right)^{-1/2} \end{aligned}$$

This proves the lower bound in (23). The upper bound follows in the same fashion. □

## 5. Proof of Theorem 3

Naturally, we would like to have an asymptotic formula for $M_b(N)$. Unfortunately, the knowledge of an asymptotic formula for the subsequence $M_b(b^n)$ does not suffice for this purpose. The situation would be different, if we had $M_b(b^n) \sim b^n$ asymptotically, since then an interpolation lemma like Lemma 2 in [6] could be used.

In order to improve on Corollary 3, we have to refine Theorem 2. To be more precise, we will be looking at a denser subsequence of $M_b(N)$. For this reason we define for positive integers $n$, $t$ and $s \leqslant t$

$$A(t, s; n) = A(t)^{n-1} \times A(s), \tag{25}$$

with

$$A(t)^0 \times A(s) = A(s).$$

Following the ideas and arguments of Section 2 with the obvious modifications, we have the rank function $r: A(t, s; n) \to N_0$ given by

$$r(\boldsymbol{a}) = \sum_{1 \leq i \leq n} a_i$$

for $\boldsymbol{a} = (a_1, \ldots, a_n) \in A(t; s; n)$, satisfying

$$r_{\max} = \max_{\boldsymbol{a} \in A(t, s; n)} r(\boldsymbol{a}) = (n-1)t + s. \tag{26}$$

The Lemmas 1 and 2 now look like the following.

**Lemma 3.** *Let $1 \leq b^* \leq b$. For $a$, $a' \in A(b^* b^{n-1} - 1)$, $a \neq a'$, let $a = a_1 + a_2 b + \cdots + a_n b^{n-1}$, $a' = a'_1 + a'_2 b + \cdots + a'_n b^{n-1}$ with $\boldsymbol{a} = (a_1, \ldots, a_n)$, $\boldsymbol{a}' = (a'_1, \ldots, a'_n) \in A(b-1, b^*-1; n)$. Then*

$$a \text{ and } a' \text{ are comparable} \Leftrightarrow \boldsymbol{a} < \boldsymbol{a}' \quad \text{or} \quad \boldsymbol{a}' < \boldsymbol{a}.$$

**Lemma 4.** *For $0 \leq l \leq r_{\max}$, the lth level number $W_l(A(t, s; n))$ satisfies*

$$W_l(A(t, s; n)) = W_{r_{\max} - l}(A(t, s; n)).$$

Using again De Bruijn's theorem on finite chain products, Lemma 4 and (26) yield the analogue of Proposition 1 for the Sperner number $s(A(t, s; n))$.

**Proposition 2.**

$$s(A(t, s; n)) = \left| \left\{ \boldsymbol{a} \in A(t, s; n) : \sum_{1 \leq i \leq n} a_i = \left[ \frac{s + (n-1)t}{2} \right] \right\} \right|.$$

For $1 \leq b^* \leq b$, we set

$$M(b, b^*; n) = M_b(b^* b^{n-1}). \tag{27}$$

By Lemma 3 and Proposition 2, we get the following.

**Corollary 4.**

$$M(b, b^*; n) = \left| \left\{ \boldsymbol{a} \in A(b-1, b^*-1; n) : \sum_{1 \leq i \leq n} a_i = \left[ \frac{b^* - 1 + (n-1)(b-1)}{2} \right] \right\} \right|.$$

Instead of (6), we now use the generating function

$$f_{n,t,s}(x) = (1 + x + \cdots + x^t)^{n-1}(1 + x + \cdots + x^s).$$

Following the argument of Section 4, we get the next lemma.

**Lemma 5.** *Let $n \geq 1$, $b \geq 2$, $1 \leq b^* \leq b$. Then*

$$M(b, b^*; n) = \frac{2}{\pi} \sum_{j=0}^{b^*-1} \left( \int_0^{\pi/2} \left( \frac{\sin bx}{\sin x} \right)^{n-1} \cos((2j - b^* + \delta)x) \, \mathrm{d}x \right),$$

*where*

$$\delta = \delta(b, b^*; n) = \begin{cases} 1 & \text{for } 2 \mid (b^* - 1 + (n-1)(b-1)), \\ 2 & \text{otherwise.} \end{cases}$$

Now we are able to prove the desired refinement of Theorem 2.

**Proposition 3.** *For fixed $b \geq 2$, $1 \leq b^* \leq b$ and $n \to \infty$, we have*

$$M(b, b^*; n) = (B\pi)^{-1/2} b^* b^{n-1} (n-1)^{-1/2} + O(b^n n^{-3/2}),$$

*where $B = \frac{1}{6}(b^2 - 1)$, and the implicit constant in $O(\ )$ only depends on $b$.*

**Proof.** Given an integer $v$, $|v| \leq b$, and $x \in I_b = (0, 2/b)$, there is a $\gamma_{10} = \gamma_{10}(x)$ satisfying $|\gamma_{10}| \leq \frac{1}{2}$ and

$$\cos vx = 1 + \gamma_{10}(vx)^2. \tag{28}$$

Combining (14) and (28) for $|v| \leq b$, we have a constant $c_3$ depending only on $b$, such that for any $x \in I_b$ there are $\gamma_{11} = \gamma_{11}(x)$ and $\gamma_{12} = \gamma_{12}(x)$ with

$$\max(|\gamma_{11}|, |\gamma_{12}|) < c_3$$

and

$$\left(\frac{\sin bx}{b \sin x}\right)^{n-1} \cos vx = e^{-B(n-1)x^2}(1 + \gamma_{11}x^2 + \gamma_{12}(n-1)x^4). \tag{29}$$

Clearly

$$\begin{aligned} \int_{2/b}^{\pi/2} \left(\frac{\sin bx}{\sin x}\right)^{n-1} \cos vx \, dx &\leq \left(\sin \frac{2}{b}\right)^{-n} \int_0^{\pi/2} dx \\ &\leq \frac{\pi}{2}\left(\frac{3b}{5}\right)^{n-1}. \end{aligned} \tag{30}$$

Applying (29), (30), (21), and (22), we get for $|v| \leq b$

$$\begin{aligned} \int_0^{\pi/2} \left(\frac{\sin bx}{b \sin x}\right)^{n-1} \cos vx \, dx &= \int_0^{2/b} + \int_{2/b}^{\pi/2} \\ &= \frac{1}{2}\left(\frac{\pi}{B(n-1)}\right)^{1/2} + O(n^{-3/2}). \end{aligned}$$

With this in mind, Lemma 5 implies the proposition. □

In order to prove Theorem 3 let $b \geq 2$ and $\varepsilon > 0$. For given $N$ there are $n$ and $1 \leq b^* < b$ such that

$$b^* b^{n-1} \leq N < (b^* + 1) b^{n-1}. \tag{31}$$

Since $M_b(N)$ is increasing, we have by (27)

$$M(b, b^*; n) \leqslant M_b(N) \leqslant M(b, b^* + 1; n).$$

For sufficiently large $N$ resp. $n$, this implies together with Proposition 3 and (31)

$$\begin{aligned} M_b(N) &> ((B\pi)^{-1/2} - \varepsilon) b^* b^{n-1} (n-1)^{-1/2} \\ &\geqslant ((B\pi)^{-1/2} - \varepsilon) b^* \frac{N}{b^* + 1} \left( \frac{\log N}{\log b} \right)^{-1/2} \end{aligned}$$

Since $b^* \geqslant 1$, this proves the lower bound in Theorem 3. The upper bound follows similarly. This finishes the proof of Theorem 3. □

It should be mentioned at this point that a further refinement of Theorem 3, using our method by considering an even denser subsequence of $M_b(N)$ than the one in Proposition 3, is impossible. The reason for this is that the ordered sets corresponding to $A(N-1)$ are chain products only for $N = b^* b^{n-1}$. Hence the theorem of De Bruijn et al. is not applicable beyond Proposition 3.

It remains, however, an interesting question, if there is a constant $C^*$ depending only on $b$, such that we have asymptotically

$$M_b(N) \sim C^* \frac{N}{(\log N)^{1/2}}$$

## References

[1] M. Aigner, Combinatorial Theory (Springer, Berlin, 1979).
[2] D. André, Mémoire sur les combinaisons régulières, Ann. Sci. École Norm. Sup. 5, Serie 2 (1876) 155–198.
[3] L. Comtet, Advanced Combinatorics, Revised and Enlarged Edition (Reidel, Dordrecht, 1974).
[4] J. Motek, Problem 86-8, The Math. Intelligencer, 8 (1986) 41.
[5] P. Montel, Sur les combinaisons avec répétitions limitées, Bull. Sci. Math. 66 (1942) 86–103.
[6] J.W. Sander, On prime powers dividing $\binom{2n}{n}$, to appear.